A minimum asymptotic mean squared error controller for an IMA(1, 1) noise process with a starting offset, and its resetting design

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Abstract

In a discrete-part manufacturing process, the noise is often described by an IMA(1, 1) process and the pure unit delay transfer function is used as the feedback controller to adjust it. The optimal controller for this process is the well-known minimum mean square error (MMSE) controller. The starting level of the IMA(1, 1) model is assumed to be on target when it starts. Considering such an impractical assumption, we adopt the starting offset. Since the starting offset is not observable, the MMSE controller does not exist. An alternative to the MMSE controller is the minimum asymptotic mean square error controller, which makes the long-run mean square error minimum.

Another concern in this article is the un-stability of the controller, which may produce high adjustment costs and/or may exceed the physical bounds of the process adjustment. These practical barriers will prevent the controller to adjust the process properly. To avoid this dilemma, a resetting design is proposed. That is, the resetting procedure in use of the controller is to adjust the process according to the controller when it remains within the reset limit, and to reset the process, otherwise.

The total cost for the manufacturing process is affected by the off-target cost, the adjustment cost, and the reset cost. Proper values for the reset limit are selected to minimize the average cost per reset interval (ACR) considering various process parameters and cost parameters. A time non-homogeneous Markov chain approach is used for calculating the ACR. The effect of adopting the starting offset is also studied here.

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1. Introduction

In manufacturing industries, two major activities in controlling the process quality is the engineering process control (EPC) and the statistical process control (SPC). The EPC has been effectively used for the process whose level wanders about the target due to the inherent noise. The wandering behavior of the noise is often described by the IMA(1, 1) model (see Box and Kramer (1992), Box and Luceño (1997), and Vander Weil (1996)). Since the noise may not be removed from the process due to some technical and/or economic reasons, the deviation of the quality characteristic from target will be forecasted and compensated by adjusting the input variable, say controller.

An efficient adjustment procedure is expected to reduce the process deviation as well as the amount of adjustment, since both the process deviation and the adjustment induce the cost. Often the effectiveness of the EPC procedure is measured in terms of the average cost incurred during operation of the process. The costs involved in the EPC procedure are the off-target
cost, the adjustment cost, and the measurement cost. The off-target cost is incurred while the process is operated off target, and the adjustment cost to adjust the process. The measurement cost includes the cost for measuring the product quality characteristics.

An efficient controller will decrease the off-target cost, but tends to increase the adjustment cost when the noise process is unstable. Thus a cost effective EPC procedure should reduce the process deviation without excessively large adjustment.

The performance of the EPC procedure fully depends on the choice of the controller. Two approaches have been used in selecting the controller: One based on the optimality criterion and the other based on the empirical use. Examples of the controller based on the optimality criterion are the minimum mean squared error (MMSE) controller, the minimum asymptotic mean squared error (MAMSE) controller and the generalized MMSE controller. The examples of the empirical use are the proportional (P) controller, integral (I) controller, and the derivative (D) controller. Often all the three controllers are called the PID controller. Although the PID controller is not derived from the optimality criterion, it has shown the robustness properties with respect to different disturbances and process dynamics. Often the PID controller corresponds to the optimality based controller. The EPC method, and the integration of the EPC and the SPC have been studied by authors such as Box, Jenkins, and Reinsel (1994), Box and Luceño (1997), Capilla, Ferrer, Romero, and Hualda (1999), Castillo (2002), Jiang and Tsui (2002), Jiang, Wu, Tsung, Nair, and Tsui (2002), Montgomery, Keats, Runger, and Messina (1994), Pan and Castillo (2003), Park (2007), Runger, Testik, and Tsung (2006) and Tsung, Shi, and Wu (1999).

The noise is usually represented by a time series model and the controller is described by a transfer function model. Since the future process deviation due to the noise can be predicted by the use of the time series model, the predicted deviation is used in determining the controller. The controller will have its own process dynamics, which should be estimated from the practical use. Usually the compensated process deviation is close to zero when an appropriate controller is used. However, the controller may not be stable and will wander away from the starting state. In such cases the adjustment cost will be excessively high and/or the controller will exceed the physical bounds, which is the maximum possible amount of the controller.

The traditional approach for the EPC procedure is to keep adjusting the process level unconditionally. When the process level deviates far from the target and an excessive adjustment is required, it would be wise to reset the process like at the beginning of the process rather than to keep adjusting the process level unconditionally. This is because an excessive adjustment incurs much cost and also the process cannot be adjusted as required due to the maximum possible amount of adjustment in the every process. It is worth to notice that the large adjustment, although it seems successful in controlling the deviation, will degrade the overall performance of the EPC procedure.

Accordingly, resetting the process means a careful retuning the process parameters as well as reinsitlizing the process conditions as is done at the beginning of a new process. Thus a reset process implies that a new process begins afterwards. At the beginning of a new process, it is assumed that one quality cycle has been completed and the next cycle begins anew as a renewal process. However, a reset process or even a new process cannot start on target always. This imperfect state of control at the starting time can be explained by adopting a starting offset.

A resetting EPC procedure is proposed to improve the inappropriateness of unconditional adjustment and to consider the reset action. The proposed procedure is to adjust the process if the magnitude of the controller is within a certain bound, and to reset the process, otherwise.

A reset cycle of the adjustment will be defined as the time from the beginning of the process to the reset signal. If the reset signal is given early, it will increase the reset cost, but will decrease the adjustment cost because it can prevent the controller becoming too large.

2. The transfer function with the noise

Suppose that the noise of a new or a reset process is well represented by an IMA(1, 1) model. Also suppose that the noise starts at an off-target level. Then the noise can be expressed as

$$Z_t = \frac{1 - \theta B}{1 - B} \varepsilon_t + S_0 \tag{1}$$

where $S_0$ denotes the starting offset, $\theta$ denotes the moving average parameter, $B$ is the backshift operator, and $\varepsilon_t$ is a white noise. We assume that $S_0 \sim N(0, \sigma^2_{s0})$, $\varepsilon_t \sim N(0, \sigma^2)$ for $t > 0$ and $\varepsilon_t = 0$ for $t \leq 0$, and $S_0$ and $\varepsilon_t$ are independent.

The model in Eq. (1) can be also expressed as the recursive relation,

$$Z_t = \begin{cases} \varepsilon_t + S_0, & t = 1 \\ Z_{t-1} + \varepsilon_t - \theta \varepsilon_{t-1}, & t > 1. \end{cases} \tag{2}$$

The random shock expression of Eq. (2) is

$$Z_t = \varepsilon_t + \lambda \sum_{j=1}^{t-1} \varepsilon_j + S_0 \tag{3}$$

where $\lambda = 1 - \theta$.

Suppose that a controller $X_t$ is used to adjust the deviation of the noise from the target, and the effect of the controller is fully observed in the next measurement. Then the total effect to the process level up to time $t$ is $\beta X_{t-1}$, where $\beta$ denotes
the process gain. A controller is sometimes called a transfer function in the sense that it describes the dynamical relation
between the controller and the quality characteristic. Such a transfer function is called the pure unit delay system or the
responsive system. The observed deviation from target after implementation of the controller will be

\[ Y_t = \beta X_{t-1} + \frac{1 - \theta B}{1 - B} \varepsilon_t + S_0. \]  

We assume \( X_0 = Y_0 = 0. \)

3. The integral controller

Consider the integral controller expressed as

\[ X_t = -\frac{d}{1 - B} Y_t \]  

for a tuning parameter \( d \). Then the observed deviation will be

\[ Y_t = -\frac{\beta d}{1 - B} Y_{t-1} + \frac{1 - \theta B}{1 - B} \varepsilon_t + S_0. \]

Thus we see the deviation of the adjusted model as

\[ Y_t = \frac{1 - \theta B}{1 - v_0 B} \varepsilon_t + \frac{1 - B}{1 - v_0 B} S_0 \]  

where \( v_0 = 1 - \beta d. \) For stationarity, we assume that the solution for \( 1 - v_0 B = 0 \) lies outside a unit circle, that is, \( |v_0| < 1. \)

For the recursive relation, the deviation can be expressed as

\[ Y_t = \begin{cases} \varepsilon_1 + S_0, & t = 1 \\ v_0 Y_{t-1} + \varepsilon_t - \theta \varepsilon_{t-1}, & t > 1. \end{cases} \]  

The mean square error (MSE) can be easily derived by the following random shock expression using the above recursive relation.

\[ Y_t = \varepsilon_t + u_0 \sum_{j=1}^{t-1} v_0^{t-j-1} \varepsilon_j + v_0^{t-1} S_0 \]  

where \( u_0 = \lambda - \beta d. \)

From Eq. (5), the adjustment, \( \nabla X_t = (1 - B)X_t \), of the integral controller is

\[ \nabla X_t = -dY_t. \]  

The MMSE controller for the process starting at on-target level (\( S_0 = 0 \)) is the integral controller with \( d = \lambda/\beta \) in Eq. (5).

Although the deviation \( Y_t \) remains on target by MMSE controller with minimum variance, the controller \( X_t \) follows a
nonstationary ARIMA(1, 1, 1). Thus it could be too expensive for adjusting the process by MMSE.

In practice, excessive manipulation of the controller is not desirable in many cases, since there may be a cost proportional
to the magnitude of the control actions. Also, there may be limitations in the range of values that the controller can take.
One way of solving such problems is to define the cost function with both the deviation and the magnitude the controller,
and try to reduce the cost.

4. The MAMSE controller

The MSE of the observed deviation is derived using its random shock expression in Eq. (8) as

\[ V(Y_t) = \left( 1 + u_0^2 \frac{1 - v_0^{2(t-1)}}{1 - v_0^2} \right) \sigma_\varepsilon^2 + v_0^{2(t-1)} \sigma_S^2. \]

Then its AMSE is

\[ \lim_{t \to \infty} V(Y_t) = \left( 1 + \frac{u_0^2}{1 - v_0^2} \right) \sigma_\varepsilon^2. \]

The minimum possible AMSE is \( \sigma_\varepsilon^2 \), and this minimum is achieved when \( u_0 = 0. \) If \( u_0 = 0 \), then we have \( d = \lambda/\beta \) and \( v_0 = \theta. \) This implies that the MAMSE controller is the integral controller with

\[ X_t = -\frac{\lambda}{\beta} \frac{1}{1 - B} Y_t. \]  

(10)
The observed deviation when the MAMSE controller is used is derived as
\[ Y_t = \epsilon_t + \frac{1 - B}{1 - \theta B} S_0 \]  \hspace{1cm} (11)
or
\[ Y_t = \begin{cases} \epsilon_1 + S_0, & t = 1 \\ \theta Y_{t-1} + \epsilon_t - \theta \epsilon_{t-1}, & t > 1 \end{cases} \]  \hspace{1cm} (12)
or
\[ Y_t = \epsilon_t + \theta^{t-1} S_0. \]  \hspace{1cm} (13)

The linear filter expression of the controller is
\[ (1 - B)X_t = -\lambda \beta \left( \epsilon_t + \frac{1 - B}{1 - \theta B} S_0 \right). \]

Since the solution of the characteristic equation, \((1 - B) = 0\), lies on the unit circle, the controller will be unstable and the adjustment cost will be excessive. Resetting the process when the controller exceeds certain bounds will prevent the controller excessively large.

The resetting procedure is to adjust the process by the controller \(X_t\) and reset the process if \(|X_t| \geq h\)
or equivalently,
\[ \left| \sum_{j=1}^{t} Y_j \right| \geq \frac{h}{d} \]  \hspace{1cm} (14)
for some given reset bound \(h\). We assume that once the process is reset, a new but the same noise process will start with a starting offset \(S_0\). Such repetition of the process is called as the renewal process.

5. The average cost per unit interval

In evaluating the effectiveness of the EPC procedure, we need to calculate the average cost per unit interval. Since the resetting procedure generates a renewal process, the average cost per unit interval can be expressed as the ratio of the expected cost per cycle and the expected cycle length (Ross, 1970). Such economic evaluation of the process control performance has been studied by authors such as Jiang and Tsui (2000), Lorenzen and Vance (1986) and Park and Reynolds (2008). For this we define a cycle of the process, which denotes the time from the start of the process to the reset signal. The cycle length or the reset time is denoted as \(T_R\). The cost parameters incurred in a cycle are the ones related to the adjustment, the off-target process level, and the resetting action.

We assume that the reset cost is fixed as \(C_R\). The off-target cost is proportional to the sum of the squared deviations \((Y_t)\). The off-target cost per unit interval is assumed as \(C_T/\sigma^2\) times the squared deviation. Similarly, the cost for each adjustment is assumed as \(C_A/\sigma^2\) times the square of each adjustment. The cost parameter \(C_T\) can be interpreted as the following. Consider an MMSE controller for a noise starting on target level without a special cause. The process deviation due to the MMSE controller will be simply a white noise with the variance \(\sigma^2\). Thus the off-target cost per unit interval for such a process will be \(C_T\).

The adjustment cost will be fixed or variable. The fixed adjustment cost case arises where adjustment might involve the stopping of a machine and/or the replacement of a tool. In that case, the adjustment cost will be fixed. For the fixed adjustment cost case, the average cost per cycle incurred due to the adjustment will be the same as the fixed adjustment cost since the process is adjusted at every interval. Thus the adjustment cost will have no effect on the choice of the optimal chart parameters.

The variable adjustment cost case arises where the adjustment cost might be proportional to the magnitude of the controller or the adjustment. In the variable adjustment cost case, it might be convenient to constrain the variability of the adjustment so that the adjustment cost may be proportional to the squared sum of the adjustment.

The total cost incurred in a cycle of the EPC procedure with the variable adjustment cost will be
\[ C_T \sum_{j=1}^{T_R} Y_j^2 + C_A \sum_{j=1}^{T_R-1} (\nabla X_j)^2 + C_R. \]

For simplicity, we define the relative cost per cycle (RCC) as the total cost divided by the off-target cost, \(C_T\), then
\[ \text{RCC} = \sum_{j=1}^{T_R} Y_j^2 + R_A \sum_{j=1}^{T_R-1} (\nabla X_j)^2 + R_R \]  \hspace{1cm} (15)
where $R_A = (C_A/C_T)$ denotes the relative adjustment cost and $R_R (= C_R/C_T)$ denotes the relative reset cost. Then the average relative cost per unit interval (ACU) is defined as

$$ACU = \frac{E(RCC)}{E(T_R)}.$$  \hfill (16)

The ACU has been derived for various values of the process and cost parameters in the next section which covers typical cases in real applications.

6. Performance of the controller

The performance of the controller is measured in terms of the ACU defined in Eq. (16). For the derivation of the ACU, we need to calculate the average time to reset (ATR) and the total MSE for a cycle. In calculating these properties, we use the Markov chain approximation.

Let the sum of the observed deviations be

$$W(t) = \sum_{j=1}^{t} Y_j,$$

then it is easily seen that the distribution of $W(t)$ given $W(t-1) = w$ depends only on $w$ and the time $t$. That is $W(t)$ constitutes a (time) non-homogeneous Markov process. The one-step transition probability from state $i$ to state $j$ at time $t$ converges in the long run. The convergence of the transition probability makes it possible to calculate the properties of the Markov chain. Details of the Markov chain approximation is in the Appendix.

In calculating the ACU values, the following process and cost parameters are selected.

- $\theta = \{0.0, 0.3, 0.6, 0.9\}$
- $\sigma = 1$, $\beta = 1$, $\sigma_S/\sigma = \{1, 3, 5\}$
- $R_R = \{0.1, 0.2, 1, 2, 5, 10, 20, 50, 100, 200, 500, 1000\}$
- $R_A = \{0.01, 0.02, 0.1, 0.2, 1, 5, 10, 50, 100\}$.

In Fig. 1, the log transformed ATR values are plotted against $h$. From the figure it is seen that the ATR increases as $h$ increases as well as $\theta$ increases. This is because the deviation becomes stable as $\theta$ increases. But the ATR looks almost the
same for different $\sigma_s$ values although it becomes unnoticeably smaller as $\sigma_s$ increases. In Fig. 2, MSE denotes the expected total MSE in a cycle, MSEB denotes the expected total MSE up to $T_r - 1$, and MSETR denotes the expected MSE at time $T_r$. Thus MSE is the sum of MSEB and MSETR.

7. Optimal choice of the reset bound

It would be helpful in practical implementation to give a guide line for the choice of the reset bound. In Fig. 3, the ACU values are plotted for a typical case ($\theta = 0.6$, $\sigma_s = 3$). In the figure each graph panel corresponds to the $R_A$ value, and in each graph panel the ACU is plotted vs. $h$ for various values of $R_A$. The vertical line with a point indicates the optimal $h$ value, which corresponds to the minimum ACU value, for the corresponding ACU curve. The case $h = 0$ means the extreme case that the process is reset at every time. The ACU for $h = 0$ is $\sigma^2 + \sigma^2_S + R_A$.

The upper four panels denote small values of $R_A$, the middle four panels medium range of $R_A$ values, and the lower four panels large values of $R_A$. In the figure when $R_A$ ranges less than or equal to 2, the optimum $h$ is very close to 0 for every $R_A$ value. This means that it would be economically efficient for resetting the process every time since the reset cost is too cheap. When $R_A$ ranges greater than or equal to 500, the ACU values are almost the same for various values of $R_A$.

When the $R_A$ value ranges less than or equal to 10, the ACU values are almost the same. When $R_A$ value ranges in the medium range, (5–100), the optimal $h$ value increases as $R_A$ increases and also as $R_A$ decreases.

As a guide for selecting $h$ value, the following parameters may be used.

i. If $R_A$ is large, we do not need to consider $R_A$ value.
ii. If $R_A$ is small, it would be recommended to reset the process every interval.
iii. If $R_A$ is in the medium range, the optimum $h$ values for different $R_A$ values can be selected accordingly from Table 1.
Fig. 3. ACU values vs. h for various values of \( R_A \) and \( R_R (\theta = 0.6, \sigma_S = 3) \). Vertical line with a point indicates h value which produces the MAMSE for the given \( R_A \) value.

Table 1
The optimum values of h for given \( R_A \) and \( R_R \) values. When \( \theta = 0.6 \) and \( \sigma_S = 3 \).

<table>
<thead>
<tr>
<th>( R_A )</th>
<th>( R_R )</th>
<th>0.1</th>
<th>0.2</th>
<th>1</th>
<th>2</th>
<th>5</th>
<th>10</th>
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<td>0.01</td>
<td>0.01 (0.126)</td>
<td>0.1 (0.216)</td>
<td>0.1 (0.933)</td>
<td>0.1 (1.83)</td>
<td>0.9 (3.69)</td>
<td>1.1 (4.41)</td>
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</tr>
<tr>
<td>0.02</td>
<td>0.01 (0.126)</td>
<td>0.1 (0.216)</td>
<td>0.1 (0.933)</td>
<td>0.1 (1.83)</td>
<td>0.9 (3.69)</td>
<td>1.1 (4.41)</td>
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</tr>
<tr>
<td>0.1</td>
<td>0.01 (0.126)</td>
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<tr>
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<td>0.01 (0.127)</td>
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<td>2.8 (30.9)</td>
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Entry in parentheses is the optimum MSE corresponding to h value.
8. Conclusions and discussions

In this article, the use of the resetting procedure is adopted to the traditional EPC procedure. The resetting procedure has been usually used in practice, but its performance has not been evaluated since most EPC procedures have been considered assuming the adjustment will be done unconditionally regardless of the magnitude of the controller. The performance of the resetting procedure has been evaluated in terms of the ACU, where each cycle is defined as the time from the start of a new process to the reset signal. Considering the adjustment cost and the reset cost, the optimum reset bound can be determined by selecting the bound which minimizes the ACU. The properties of the EPC procedure have been calculated by the use of the time non-homogeneous Markov property of the observed deviation.

It has been shown that the resetting procedure is a cost effective one when compared to the traditional no-rest procedure. The starting offset does not show any significant influence to the properties of the procedure unless its variance is extremely large.

For further research, it would be more practically useful if the adjustment cost can be expressed in terms of other than the adjustment (that is, one time amount of adjustment), for example, such as the fixed cost and the controller (that is, the total amount of the adjustment).

Appendix

The Markov chain approach for deriving the average time to reset (ATR) and the mean squared error (MSE) can be explained as follows.

The real line that the controller can take is classified as the transient region and the absorbing region. Then the transient region \((-h/d, h/d)\) is divided into \(n_T\) subintervals of equal width, \(w = 2h/(dn_T)\), and each subinterval is treated as a transient state of a Markov chain, \(\{S_{T,i}, \ i = 1, 2, \ldots, n_T\}\). Each transient state of the Markov chain has the corresponding midpoints as \(\{m_{T,i}, \ i = 1, 2, \ldots, n_T\}\), and the lower and upper limits as \(\{l_{T,i}, u_{T,i}\}, \ i = 1, 2, \ldots, n_T\). The absorbing region is \(R = (-\infty, -h/d) \cup (h/d, \infty)\), but only a sub-region of \(R\) is considered for calculation. The sub-region, \((-h/d - wn_T/2, -h/d) \cup (h/d, h/d + wn_T/2)\), is divided into \(n_A\) number of subintervals with width \(w\), and each subinterval is treated as an absorbing state of the Markov chain. Similarly to the transient states, each absorbing state has the corresponding midpoints as \(\{m_{A,i}, \ i = 1, 2, \ldots, n_A\}\), and the lower and upper limits as \(\{l_{A,i}, u_{A,i}\}, \ i = 1, 2, \ldots, n_A\).

The transient state transition matrix from time \(t - 1\) to time \(t\) is denoted as

\[
Q_t = [q_{ij,t}]_{n_T \times n_T} \tag{A.1}
\]

where \(q_{ij,t}\) denotes the transition probability from state \(S_i\) at time \(t - 1\) to state \(S_j\) at time \(t\). This transition probability is approximated as

\[
q_{ij,t} = P \left( \sum_{l=1}^{t} Y_l = m_{T,j} \mid \sum_{l=1}^{t-1} Y_l = m_{T,i} \right) \\
\approx \Phi \left( \frac{u_{T,j} - m_{T,i}}{v_t} \right) - \Phi \left( \frac{l_{T,j} - m_{T,i}}{v_t} \right)
\]

where \(v_t\) denotes the standard deviation of \(\sum_{l=1}^{t} Y_l\).

Now, let \(T(S_i)\) denote the time that \(W(t)\) visits the state \(S_i\), and \(V(S_i)\) denote the number of times that \(W(t)\) visits the state \(S_i\), not including the starting state. Also define the indicator function,

\[
l_t \{S_i\} = \begin{cases} 
1, & \text{if } W(t) \text{ visits } S_i \text{ at time } t, \\
0, & \text{o.w.}
\end{cases}
\]

Then we can easily obtain the following.

\[
V(S_{T,i}) = \sum_{t=1}^{\infty} l_t \{S_{T,i}\} \\
E[V(S_{T,i})] = \sum_{t=1}^{\infty} P(T(S_{T,i}) = t) \\
E[T_G - 1] = \sum_{i=1}^{n_T} E[V(S_{T,i})] \tag{A.2}
\]

Let \(s_0' = (0, \ldots, 0, 1, 0, \ldots, 0)\) be the starting state vector where the middle point (0 is included) is 1 and all the others are 0. Also let \(1_{(i)} = (0, \ldots, 0, 1, 0, \ldots, 0)\) be a vector whose \(i\)-th element is 1 and all the others are 0. Then we have

\[
P(T(S_{T,i}) = t) = \mathbf{s}_0' \cdot Q_i \cdot Q_2 \cdots Q_t \cdot 1_{(i)} \\
= \mathbf{s}_0' Q_t 1_{(i)}
\]
where $Q^*_i = Q_1 \cdot Q_2 \cdots Q_i$. From Eq. (A.2), we have

$$E(T_R) = \sum_{i=1}^{\infty} s_0^i Q^*_i 1 + 1$$

(A.3)

where $1$ is a unit vector with the corresponding dimension.

For evaluations of the properties related to the MSE, it is necessary to derive the value of the Markov chain at the reset time. Let $T(S_i, S_j)$ denote the time that $W(t - 1)$ visits the state $S_i$ and $W(t)$ visits the state $S_j$, and $V(S_i, S_j)$ denote the number of times that $W(t - 1)$ visits the state $S_i$ and $W(t)$ visits the state $S_j$, not including the starting state. Also define the indicator function,

$$I_t(S_i, S_j) = \begin{cases} 1, & \text{if } W(t - 1) \text{ visits } S_i \text{ and } W(t) \text{ visits } S_j, \\ 0, & \text{o.w.} \end{cases}$$

Then we can easily obtain the following similarly as before.

$$V(S_i, S_j) = \sum_{t=1}^{\infty} I_t(S_i, S_j)$$

$$E \{ V(S_i, S_j) \} = \sum_{t=1}^{\infty} P(T(S_i, S_j) = t)$$

$$P(T(S_i, S_j) = t) = \begin{cases} s_0^i Q^*_i - 1 1_{(i)} \cdot 1_{(j)} Q^*_i 1_{(j)}, & \text{if both } S_i, S_j \text{ are transient} \\ s_0^i Q^*_i - 1 1_{(i)} R_i 1_{(j)}, & \text{if } S_i \text{ is transient and } S_j \text{ is absorbing} \end{cases}$$

The MSE can be expressed as the sum of the MSE immediately before the reset time (MSEB) and the MSE at the reset time (MSETR). The MSE is derived as

$$E \left( \sum_{t=1}^{T_R - 1} Y_t^2 \right) = E \left( \sum_{t=1}^{T_R - 1} (W(t) - W(t - 1))^2 \right)$$

$$= \sum_{i=1}^{n_T} \sum_{j=1}^{n_T} (m_{T,j} - m_{T,i})^2 V(S_{T,i}, S_{T,j})$$

$$= \sum_{i=1}^{n_T} \sum_{j=1}^{n_T} (m_{T,j} - m_{T,i})^2 E \{ V(S_{T,i}, S_{T,j}) \}$$

$$= \sum_{i=1}^{n_T} \sum_{j=1}^{n_T} (m_{T,j} - m_{T,i})^2 \sum_{t=1}^{\infty} P(T(S_{T,i}, S_{T,j}) = t)$$

$$= \sum_{i=1}^{n_T} \sum_{j=1}^{n_T} \sum_{t=1}^{\infty} (m_{T,j} - m_{T,i})^2 \cdot s_0^i Q^*_i - 1 1_{(i)} \cdot 1_{(j)} Q^*_i 1_{(j)}$$

$$= \sum_{i=1}^{\infty} s_0^i Q^*_i - 1 \sum_{i=1}^{n_T} \sum_{j=1}^{n_T} 1_{(i)} Q^*_i 1_{(j)} (m_{T,j} - m_{T,i})^2$$

$$= \sum_{i=1}^{\infty} s_0^i Q^*_i - 1 Q^*_i 1$$

(A.4)

where $Q^*_i = [q_{ij}(t) (m_{T,j} - m_{T,i})^2]_{n_T \times n_T}$.

For derivation of MSETR, define the transition probability from a transient state to an absorbing state as

$$r_{ij} = P \left( \sum_{l=1}^{t} Y_l = m_{A,j} \left| \sum_{l=1}^{t-1} Y_l = m_{T,i} \right. \right)$$

$$\approx \Phi \left( \frac{u_{A,j} - m_{T,i}}{v_t} \right) - \Phi \left( \frac{l_{A,j} - m_{T,i}}{v_t} \right)$$

and let

$$R_i = [r_{ij}(t) (m_{A,j} - m_{T,i})^2]_{n_T \times n_A}$$

(A.5)

Then the MSETR is derived as
Summarizing the results from the Markov chain approach, we have the following expressions.

\[ E(Y_{TR}^2) = E((W(T_R) - W(T_R - 1))^2) \]
\[ = E \left( \sum_{i=1}^{n_T} \sum_{j=1}^{n_A} (m_{A,j} - m_{T,i})^2 V(S_{T,i}, S_{A,j}) \right) \]
\[ = \sum_{i=1}^{n_T} \sum_{j=1}^{n_A} (m_{A,j} - m_{T,i})^2 E(V(S_{T,i}, S_{A,j})) \]
\[ = \sum_{i=1}^{n_T} \sum_{j=1}^{n_A} \sum_{t=1}^{n_T} (m_{A,j} - m_{T,i})^2 E(T(S_{T,i}, S_{A,j}) = t) \]
\[ = \sum_{i=1}^{n_T} \sum_{j=1}^{n_A} \sum_{t=1}^{n_T} (m_{A,j} - m_{T,i})^2 \cdot s_i^t Q_{s-1}^t 1_{(i)} \cdot 1_{(j)} 1_R 1_{(j)} \]
\[ = \sum_{i=1}^{n_T} s_i^t Q_{s-1}^t 1_{(i)} \sum_{t=1}^{n_T} 1_{(i)} 1_R 1_{(j)} (m_{A,j} - m_{T,i})^2 \]
\[ = \sum_{i=1}^{n_T} s_i^t Q_{s-1}^t 1_R 1 \]  \hspace{1cm} (A.6)

where \( R_q^f = [q_{ij}, (m_{A,j} - m_{T,i})^2]_{n_T \times n_A} \).

Since \( Q_s \) converges as \( t \to \infty \), let \( \lim_{t \to \infty} Q_s = Q \). Let \( r \) be the time that
\[ Q_s \approx Q, \quad Q^f \approx Q^5, \quad R_q^f \approx R^5. \]

Then we have
\[ \sum_{t=1}^{\infty} s_i^t Q_{s-1}^t = \sum_{t=1}^{r-1} s_i^t Q_{s-1}^t + s_i^t Q_{s-1}^{r-1} \sum_{t=r}^{\infty} Q_{s-1}^{t-r+1} \]
\[ = \sum_{t=1}^{\infty} s_i^t Q_{s-1}^t + s_i^t Q_{s-1}^{r-1} Q(I - Q)^{-1}. \]  \hspace{1cm} (A.7)

Thus we have the following approximations using Eq. (A.7) and Eqs. (A.3), (A.4) and (A.6).

\[ ATR \approx \sum_{t=1}^{r-1} s_i^t Q_{s-1}^t 1 + s_i^t Q_{s-1}^{r-1} Q(I - Q)^{-1} 1 + 1 \]
\[ MSEB \approx \sum_{t=1}^{r-1} s_i^t Q_{s-1}^t Q^5 1 + s_i^t Q_{s-1}^{r-1} (I - Q)^{-1} Q^5 1 \]
\[ MSCTR \approx \sum_{t=1}^{r-1} s_i^t Q_{s-1}^{r-1} S_t + s_i^t Q_{s-1}^{r-1} (I - Q)^{-1} R^5 1. \]

Summarizing the results from the Markov chain approach, we have the following expressions.

\[ E(T_R) = ATR \]
\[ \sum_{j=1}^{T_R} Y_j^2 = MSEB + MSCTR \]
\[ \sum_{j=1}^{T_R-1} (\nabla X_j)^2 = \Delta^2 \sum_{j=1}^{T_R-1} Y_j^2 = \Delta^2 MSEB. \]

References


